

# Chapter 4

## Hydrostatics

### 4.1 Fluids

One of the first questions we need to explore is— what is a fluid ? Or we might ask— what is the difference between a solid and a fluid ? We have a general, vague idea of the difference. A solid is “hard” and not easily deformed, whereas a fluid is “soft” and is easily deformed (we can easily move through air). Although quite descriptive, these casual observations of the differences between solids and fluids are not very satisfactory from a scientific or engineering point of view. A more specific distinction is based on how materials deform under the action of an external load. *A fluid is defined as a substance that deforms continuously when acted on by a shearing stress of any magnitude.* A shearing stress (force per unit area) is created whenever a tangential force acts on a surface. When common solids such as steel or other metals are acted on by a shearing stress, they will initially deform (usually a very small deformation), but they will not continuously deform (flow). However, common fluids such as water, oil, and air satisfy the definition of a fluid—that is, they will flow when acted on by a shearing stress. Some materials, such as slurries, tar, putty, toothpaste, and so on, are not easily classified since they will behave as a solid if the applied shearing stress is small, but if the stress

exceeds some critical value, the substance will flow. These fluids are called *Bingham plastic*.

Although the molecular structure of fluids is important in distinguishing one fluid from another, it is not possible to study the behavior of individual molecules when trying to describe the behavior of fluids at rest or in motion. Rather, we characterize the behavior by considering the average, or macroscopic, value of the quantity of interest, where the average is evaluated over a small volume containing a large number of molecules. We thus assume that all the fluid characteristics we are interested in (pressure, velocity, etc) vary continuously throughout the fluid—that is, we treat the fluid as a *continuum*.

## Newtonian Fluid

A fluid whose viscosity ( $\mu$ ) does not change with applied shear rate ( $\dot{\gamma}$ ) is called a Newtonian fluid. It is to be noted that viscosity of such fluids could change with temperature and pressure but it is independent of applied shear rate. Referring to § 1.12, for Newtonian fluids the shear stress is proportional to the velocity gradient (also known as shear rate) :

$$\tau_{xy} = \mu \frac{dv_x}{dy}$$

and the constant of proportionality is the viscosity of a Newtonian fluid which is independent of shear rate. Thus according to Newton's law of viscosity presented in § 1.12, a plot of  $\tau_{xy}$  versus  $dv_x/dy$  for a Newtonian fluid should give a straight line through the origin, and the slope of this line is the viscosity of the fluid at the given temperature and pressure. The ratio of viscosity ( $\mu$ ) to density ( $\rho$ ) of a fluid is called *kinematic viscosity*:  $\nu \equiv \mu/\rho$ . Typical units of  $\mu$  and  $\nu$  are [ $\text{g cm}^{-1} \text{sec}^{-1}$ ] and [ $\text{cm}^2 \text{sec}^{-1}$ ], respectively.

## non-Newtonian Fluid

There are a few industrially important fluids whose viscosity change with applied shear rate and they are termed as non-Newtonian fluids. For these fluids a plot of  $\tau_{xy}$  versus

$dv_x/dy$  would not give a straight line through the origin. For these fluids the steady-state rheological behavior can be expressed by a generalized form:

$$\tau_{xy} = \eta \frac{dv_x}{dy}$$

where  $\eta$  may be expressed as a function of either shear rate ( $dv_x/dy$ ) or shear stress ( $\tau_{xy}$ ). In regions in which  $\eta$  decreases with increasing shear rate the behavior is termed *pseudoplastic* or *shear thinning*; in regions in which  $\eta$  increases with increasing shear rate the behavior is termed *dilatant* or *shear thickening*. Ofcourse, if  $\eta$  is independent of shear rate, the behavior is Newtonian, with  $\eta = \mu$ .

A *Bingham plastic* remains rigid when the shear stress is of smaller magnitude than the yield stress  $\tau_0$  but flows somewhat like a Newtonian fluid when the shear stress exceeds  $\tau_0$ .

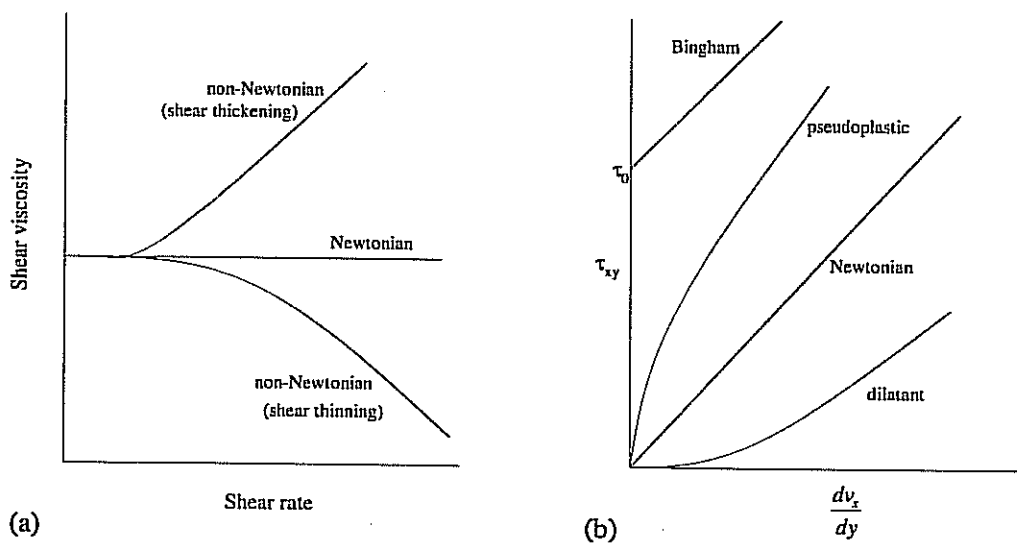


Figure 4.1: Summary of Newtonian and non-Newtonian fluids.

## 4.2 Compressibility of Fluids

An important question to answer when considering the behavior of a particular fluid is how easily can the volume (and thus the density) of a given mass of the fluid be changed when there is a change in pressure? That is, how compressible is the fluid? A property that is commonly used to characterize compressibility is the *bulk modulus*,  $E_v$ , defined as

$$E_v = -\frac{dp}{dV/V}$$

where  $dp$  is the differential change in pressure needed to create a differential change in volume,  $dV$ , of a volume  $V$ . The negative sign is included since an increase in pressure will cause a decrease in volume. Since a decrease in volume of a given mass,  $m = \rho V$ , will result in an increase in density, above equation can also be expressed as

$$E_v = \frac{dp}{d\rho/\rho}$$

Large values for the bulk modulus indicate that the fluid is relatively incompressible—that is, it takes a large pressure change to create a small change in volume. Since such large pressures are required to effect a change in volume, we conclude that liquids can be considered as *incompressible* for most practical engineering applications. Thus for these fluids densities remain almost constant over wide range of pressure and temperature.

The effects of compressibility, however, are more a property of the situation (read flow conditions) than of the fluid itself. For example, the flow of air (which is a compressible fluid) at low velocities is described by exactly the same equations that describe the flow of water (which is an incompressible fluid). Thus, instead of being classified according to the fluid, compressibility effects are considered a property of the flow. In other words, an incompressible *flow* does not necessarily imply that the *fluid* itself is incompressible.

### 4.3 Forces Acting on a Fluid

Forces acting on a fluid are divided into two general groups: (i) body forces and (ii) surface forces. Body forces are those which act without physical contact, for example, gravity and electrostatic forces. On the other hand, pressure, stresses, and frictional forces require physical contact for transmission. Since a surface is required for the action of these forces they are called surface forces. Stress is therefore a surface force per unit area.

Consider two sections of a fluid separated by an arbitrary surface. Further, focus on a local region on the interface whose area is  $\Delta A$ . The force exerted by the top section on the bottom section is denoted by  $\Delta \mathbf{F}$  which is called as the contact force. One can then define an intensity of the contact force as the force divided by the area of contact. If the area tends to zero, we have the local value of the intensity of the contact force, also called as the traction at that point.

#### Stress at a Point

Consider the force  $\Delta \mathbf{F}$  acting on an element  $\Delta A$  of a fluid interface. The force  $\Delta \mathbf{F}$  is resolved into components normal and parallel to the surface of the element. The force per unit area or stress at a point is defined as the limit of  $\Delta \mathbf{F}/\Delta A$  as  $\Delta A \rightarrow \delta A$ , where  $\delta A$  is the smallest area for which continuum assumption holds.

$$\lim_{\Delta A \rightarrow \delta A} \frac{\Delta F_n}{\Delta A} = \sigma_{ii} \qquad \lim_{\Delta A \rightarrow \delta A} \frac{\Delta F_s}{\Delta A} = \tau_{ij}$$

Here  $\sigma_{ii}$  is called the normal stress and  $\tau_{ij}$  the shear stress. For a static fluid (no motion of fluid) the shear stress is zero.

#### Surface Tension

Surface tension is a quantity which we often deal with in daily life without realizing it. It plays a role in washing and cleaning procedures as well as in lubricants as used in

automobiles and cosmetics. The only reason the Water Beetle does not drown is because it is being held up by surface tension.

Surface tension is created through the attraction of the molecules in liquids to each other. In one views a molecule at the interior of a medium, it would be equally attracted by all neighboring molecules. The effect is that it is attracted to all sides with the same force, so that the resulting force is zero. On the other hand, if a molecule is located at the surface of a liquid, the attraction from the interior of the medium works further on the one side, whereas there are no more molecules from the other side. Hence the resulting force is directed towards the interior of the liquid. On the microscopic scale this causes drops of liquid to be round since the surface of the liquid is being minimized that way.

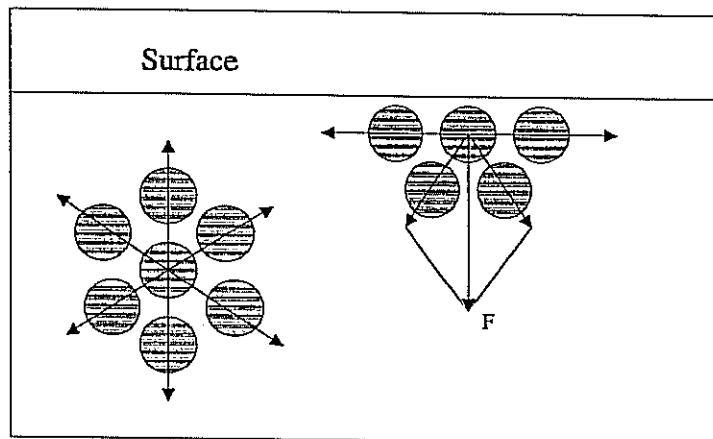


Figure 4.2: Cause of surface tension.

The surface of the liquid is under tension and behaves like a stretched membrane or rubber band. The intensity of the molecular attraction per unit length along any line in

the surface is called the *surface tension* and is denoted by  $\sigma$ .

Table 4.1: Surface tension of common liquids in contact with air

Water	0.072 N/m
Benzene	0.209 N/m
Methyl alcohol	0.023 N/m
Mercury	4.5 N/m

## Contact Angle

When a liquid is in contact with a solid, there are <sup>adhesive</sup> adhesive forces acting at solid-liquid interface. If these adhesive forces are larger than the cohesive forces between the liquid molecules then the liquid tends to wet the surface. The liquid forms a concave meniscus at the solid wall. Water in contact with a glass is an example of such situation. Although it is difficult to measure or determine the cohesive and adhesive forces, the contact angle itself can be measured. The contact angle is an indication of the relative strength of the adhesive and cohesive forces. For a liquid that wets a solid surface, the contact angle must be less than  $90^\circ$ . When the adhesive forces are smaller than cohesive forces liquid does not wet the surface and forms a convex surface. Mercury in glass is an example of such a fluid.

Among common phenomena associated with surface tension is the rise (or fall) of a liquid in a capillary tube. If a small open tube is inserted into water, the water level in the tube will rise above the water level outside the tube as is illustrated in Figure 4.3. In this situation we have a liquid-gas-solid interface. For the case illustrated there is an attraction (adhesion) between the wall of the tube and liquid molecules, which is strong enough to overcome the mutual attraction (cohesion) of the molecules and pull them up

to the wall. Hence, the liquid is said to *wet* the solid surface.

The height,  $h$ , is governed by the value of the surface tension,  $\sigma$ , the tube radius,  $R$ ,

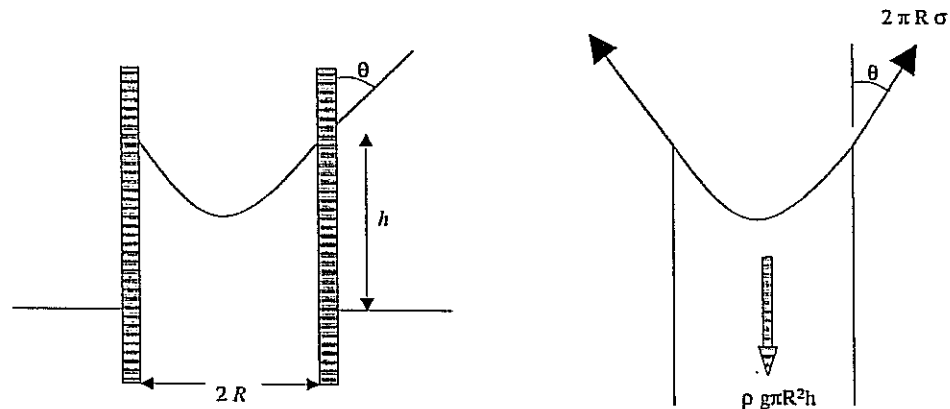


Figure 4.3: Effect of capillary action in small tubes.

the density of the liquid,  $\rho$ , and the *angle of contact*,  $\theta$ , between the fluid and tube. From the free-body diagram shown in Figure 4.3 we see that the vertical force due to the surface tension is equal to  $2\pi R\sigma \cos \theta$  and the weight is  $\rho g\pi R^2 h$  and these two forces must balance for equilibrium. Thus

$$\rho g\pi R^2 h = 2\pi R\sigma \cos \theta$$

so that the height is given by the relationship

$$h = \frac{2\sigma \cos \theta}{\rho g R}$$

### Example 1: Drop hanging on a pipette

The problem is shown schematically in Figure 4.4. Calculate the maximum size of the drop. The net force due to surface tension acts over a perimeter of  $\pi d_c$  and acts in the direction shown in the figure. The magnitude of this force is equal to  $\pi d_c \sigma$ . The



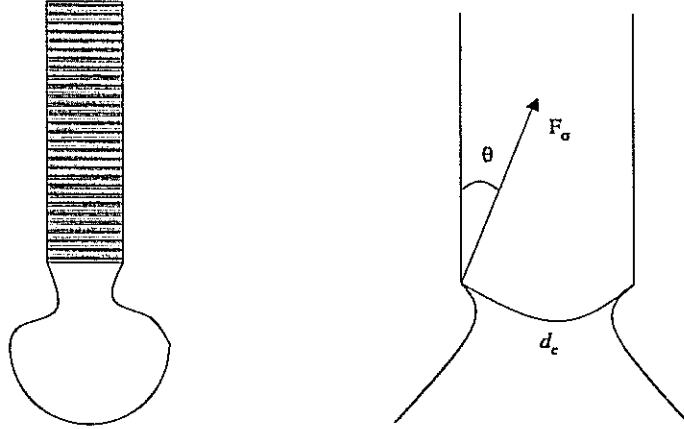


Figure 4.4: Drop hanging on the tip of a capillary.

force has a vertical component equal to  $\sigma \pi d_c \cos \theta$ . For the stable condition (drop just hanging in there) this has to be balance by the downward force due to gravity . The latter is given as:

$$\text{Gravity Force} = \pi d_d^3 \rho g / 6$$

where  $d_d$  is the radius of the drop.

The criteria for equilibrium can therefore be stated as:

$$\pi d_c \sigma \cos \theta = \pi d_d^3 \rho g / 6$$

Maximum value of the LHS occurs when  $\theta = 0$ . Hence the maximum, drop radius can be written as:

$$d_d^3_{\text{max}} = 6 d_c \sigma / (\rho g)$$

The equation can be rearranged into a dimensionless form by defining a dimensionless group (the Bond number) as:

$$Bo = \frac{d_c^2 \rho g}{\sigma}$$

The physical significance of Bond number is that it represents the ratio of gravity force to the surface tension force acting on a drop. The equation for the maximum drop size in terms of dimensionless group is therefore given by:

$$\frac{d_d}{d_c} = (6)^{1/3} B_o^{-1/3} = 1.82 B_o^{-1/3}$$

## Capillary Hydrostatics: The Young-Laplace Equation

Here we derive a very basic and important equation related to the physics of fluid interfaces. We want to show that interfacial tension can give rise to a pressure difference across an interface that separates a pair of immiscible fluids. In addition, we want to show how that pressure difference can be calculated from the shape and size of the interface.

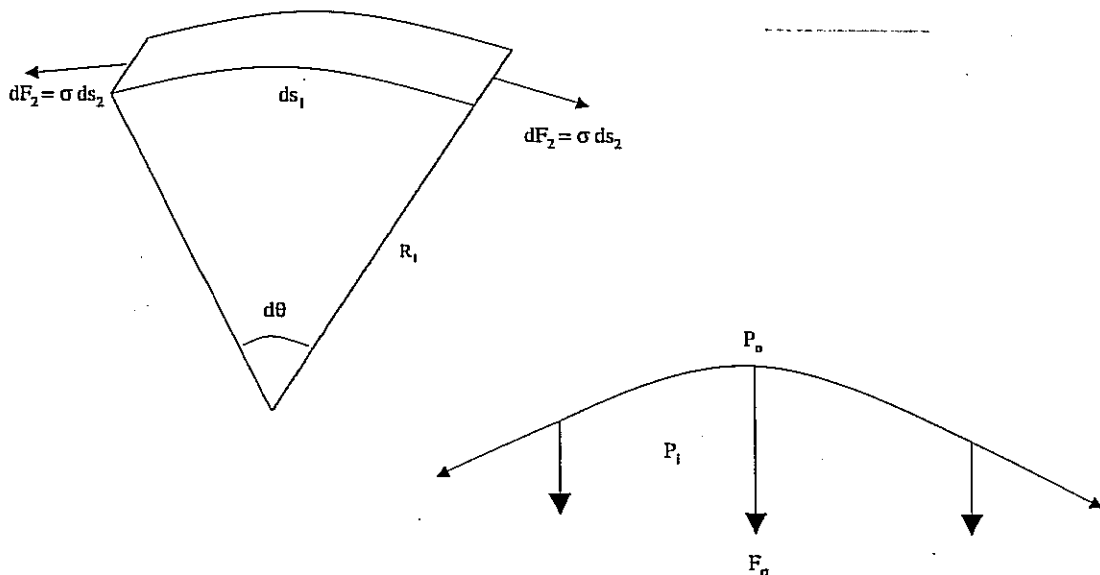


Figure 4.5: Sketch for the force balance on the area separating two immiscible fluids.

Consider the force balance on the area separating two immiscible fluids as shown in Figure 4.5. Along each of the lines  $ds_2$  is a force  $\sigma ds_2$  tangent to the surface. The

vertical component of each force is simply  $(\sigma ds_2) \sin(d\theta/2)$ . The angle  $d\theta$  is related to the arc  $ds_1$  through the radius of curvature of the arc  $R_1$ :

$$ds_1 = R_1 d\theta$$

For *differential* (i.e., *small*) *angles*, a good approximation to the sine function is

$$\sin\left(\frac{1}{2}d\theta\right) = \frac{1}{2}d\theta$$

Using these results, and summing the *two* forces along the *pair* of  $ds_2$  lines, we find

$$dF_2 = \frac{\sigma}{R_1} ds_1 ds_2$$

By identical argument, the vertical component of forces acting along the pairs of  $ds_1$  lines is

$$dF_1 = \frac{\sigma}{R_2} ds_1 ds_2$$

Thus the net vertical component of force due to interfacial tension is the sum

$$dF_\sigma = \sigma ds_1 ds_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

Note that this force is in the downward direction. If the fluids are *static*, then the only stresses on either side of the surface separating the two fluids are static pressures, say  $P_o$  and  $P_i$ , where  $P_i$  is the pressure on the concave side ("inside") of the interface. These pressures also give rise to forces in the vertical direction of magnitudes  $P_o ds_1 ds_2$  and  $-P_i ds_1 ds_2$ . Hence, at equilibrium, the force balance on the differential surface becomes (taking forces as positive down)

$$\left[ P_o - P_i + \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \right] ds_1 ds_2 = 0$$

But this results is valid for *any* area  $ds_1 ds_2$  as long as it is a differential area. Hence it follows that the bracketed term vanishes on any small region of the surface. Therefore the above equation gives the condition of equilibrium of normal stresses across a static interface separating a pair of immiscible fluids:

$$P_i - P_o = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

This relation is known as the Young-Laplace equation. Physically the equation tells us that interfacial tension causes an increased pressure on the “inside” of a surface, the magnitude depending on the radii or curvature of the surface.

#### Example 2: Pressure inside a gas bubble

A spherical gas bubble of 1 mm radius is inside a liquid. Calculate the pressure inside the bubble relative to the pressure outside.

For a spherical bubble  $R_1 = R_2 = 1 \times 10^{-3}$  mm.  $\sigma = 0.072$  N/m for water. Using the Laplace-Young equation  $\Delta P = 0.072 \frac{2}{1 \times 10^{-3}} = 144$  N/m<sup>2</sup> higher inside the bubble. The difference in pressure is 0.0014 atm but this difference can have a significant effect on the fluid dynamics.

## 4.4 Pressure at a Point

The term pressure is used to indicate the normal force per unit area at a given point acting on a given plane within the fluid mass of interest. For a static fluid, the normal stress at a point may be determined from the application of Newton’s laws to a fluid element as the fluid element approaches zero size. Note that there can be no shearing stress in a static fluid. Thus the only surface forces present will be those due to normal stresses. Now a question that immediately arises is how the pressure at a point varies with the orientation of the plane passing through the point. To answer this question, consider the free-body diagram as shown in Figure 4.6, that was obtained by removing a small triangular wedge of fluid from some arbitrary location within a fluid mass. Since we are considering the situation in which there are no shearing stresses, the only external forces acting on the wedge are due to the normal stresses and the weight.

For a body at rest,  $\sum \mathbf{F} = 0$ . In the  $x$  direction this gives

$$\Delta F_x - \Delta F_s \sin \theta = 0$$

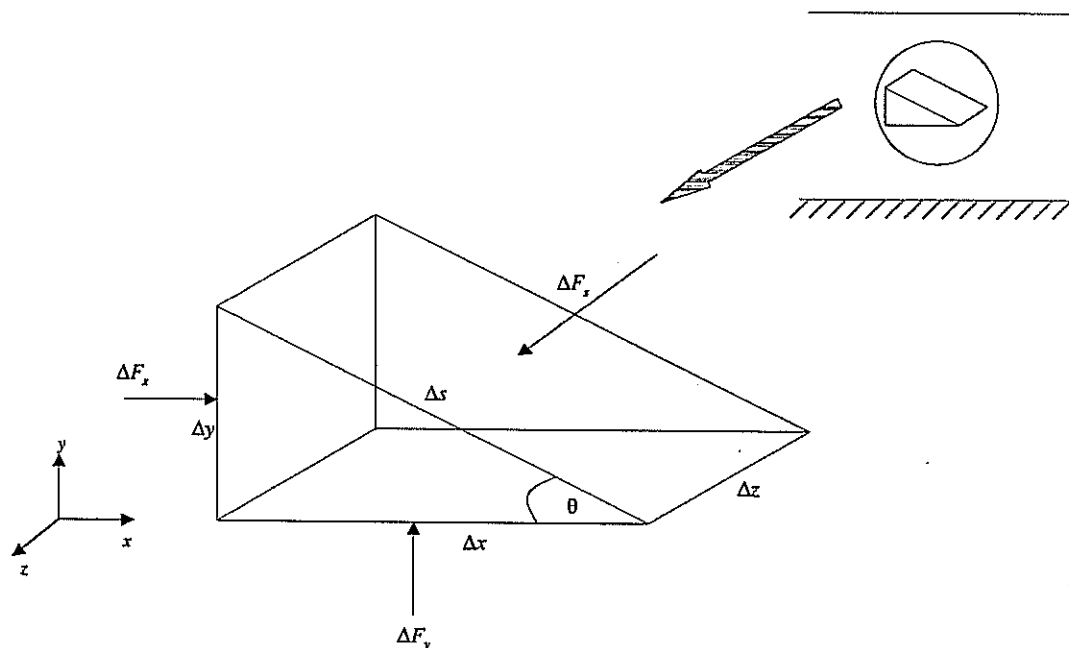


Figure 4.6: Element in a static fluid.

Since  $\sin \theta = \Delta y / \Delta s$ , the above equation becomes

$$\Delta F_x - \Delta F_s \frac{\Delta y}{\Delta s} = 0$$

Dividing through by  $\Delta y \Delta z$  and taking the limit as the volume of the element approaches zero, we obtain

$$\lim_{\Delta V \rightarrow 0} \left[ \frac{\Delta F_x}{\Delta y \Delta z} - \frac{\Delta F_s}{\Delta s \Delta z} \right] = 0$$

Noting that normal stress is positive in tension while it is negative in compression, we obtain, by evaluating the above equation

$$\begin{aligned} -\sigma_{xx} - (-\sigma_{ss}) &= 0 \quad \text{or} \\ \sigma_{xx} &= \sigma_{ss} \end{aligned} \tag{4.1}$$

Similarly in the  $y$  direction,  $\sum F = 0$  yields

$$\Delta F_y - \Delta F_s \cos \theta - \rho g \frac{\Delta x \Delta y \Delta z}{2} = 0$$

where  $\rho g \frac{\Delta x \Delta y \Delta z}{2}$  is the weight of the fluid element. Since  $\cos \theta = \Delta x / \Delta s$ , the above equation becomes

$$\Delta F_y - \Delta F_s \frac{\Delta x}{\Delta s} - \rho g \frac{\Delta x \Delta y \Delta z}{2} = 0$$

Dividing through by  $\Delta x \Delta z$  and taking the limit as before, we obtain

$$\lim_{\Delta V \rightarrow 0} \left[ \frac{\Delta F_y}{\Delta x \Delta z} - \frac{\Delta F_s}{\Delta s \Delta z} - \frac{\rho g \Delta y}{2} \right] = 0$$

which becomes

$$\begin{aligned} -\sigma_{yy} - (-\sigma_{ss}) - \frac{\rho g}{2}(0) &= 0 \quad \text{or} \\ \sigma_{yy} &= \sigma_{ss} \end{aligned} \tag{4.2}$$

Combining Equations (4.1) and (4.2) we get

$$\sigma_{ss} = \sigma_{xx} = \sigma_{yy} \tag{4.3}$$

The angle  $\theta$  was arbitrary chosen and it does not appear in Equation (4.3) so we conclude that the normal stress at a point in a static fluid is independent of direction, and is therefore a scalar quantity.

Similarly one can easily show that *the pressure at a point in a fluid at rest, or in motion, is independent of direction as long as there are no shearing stresses present.* This important result is known as *Pascal's law*.

Since the fluid element is at rest, the only surface forces acting are those due to the normal stress. If we were to measure the force per unit area acting on a submerged element, we would observe that it acts inward or to place the element in compression. The quantity measured is, of course, pressure, which in light of the preceding development, must be the negative of the normal stress. This important simplification, the reduction of stress, a tensor, to pressure, a scalar, may also be shown for the case of zero shear stress in a flowing fluid. When shearing stresses are present, the normal stress components at a point may not be equal; however, the pressure is still equal to the average

normal stress; that is

$$P = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

## 4.5 Pressure Variation in a Static Fluid

Although we have answered the question of how the pressure at a point varies with direction, we are now faced with an equally important question- how does the pressure in a fluid in which there are no shearing stresses vary from point to point?

We will be carrying out our analysis in a *non-inertial* frame of reference. A *non-inertial* frame of reference is a set of coordinate system which has some significant absolute acceleration of its own. For example a frame of reference attached to a moving railroad tank car would be classified as a *non-inertial* frame of reference. In this frame of reference Newton's second law states that sum of the external forces is equal to the product of the mass and its acceleration, i.e.,  $\sum \mathbf{F} = ma$ . On the other hand, an *inertial* frame of reference has negligible absolute acceleration of its own. For example, a frame of reference attached to earth would be classified as an *inertial* frame of reference since it is safe to neglect absolute acceleration of earth ( $\approx 0.03$  m/s). In this case  $\sum \mathbf{F} = 0$ .

In any case, we will be carrying out our analysis in a *non-inertial* frame of reference which can be reduced to the case of an *inertial* frame of reference simply by setting  $a = 0$  in our final results.

Consider a small rectangular element of fluid removed from some arbitrary position within the mass of fluid of interest attached to a *non-inertial* frame of reference as shown in Figure 4.7. There are two types types of forces acting on this element: *surface forces* due to the pressure, and a *body force* equal to the weight of the element. Since the fluid is at rest there are no shear stresses acting on this fluid element. From Figure 4.7 it is clear that:

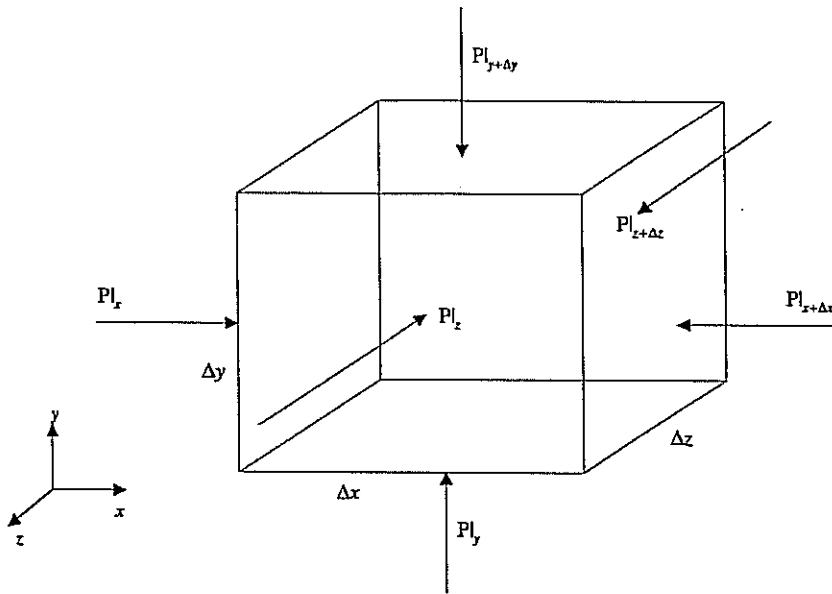


Figure 4.7: Pressure forces on a static fluid element.

$(P|_x - P|_{x+\Delta x})\Delta y \Delta z$  is the net pressure force in the  $x$  direction

$(P|_y - P|_{y+\Delta y})\Delta x \Delta z$  is the net pressure force in the  $y$  direction

$(P|_z - P|_{z+\Delta z})\Delta x \Delta y$  is the net pressure force in the  $z$  direction

$\rho g(\Delta x \Delta y \Delta z)$  is the force acting on the element due to gravity

Thus  $\sum \mathbf{F} = m\mathbf{a}$  gives

$$\begin{aligned} \rho \mathbf{g}(\Delta x \Delta y \Delta z) &+ (P|_x - P|_{x+\Delta x})\Delta y \Delta z \mathbf{e}_x \\ &+ (P|_y - P|_{y+\Delta y})\Delta x \Delta z \mathbf{e}_y \\ &+ (P|_z - P|_{z+\Delta z})\Delta x \Delta y \mathbf{e}_z \\ &= \rho \mathbf{a}(\Delta x \Delta y \Delta z) \end{aligned}$$

Dividing by the volume of the element  $(\Delta x \Delta y \Delta z)$ , we see that the above equation becomes

$$\rho \mathbf{g} - \frac{P|_{x+\Delta x} - P|_x}{\Delta x} \mathbf{e}_x - \frac{P|_{y+\Delta y} - P|_y}{\Delta y} \mathbf{e}_y - \frac{P|_{z+\Delta z} - P|_z}{\Delta z} \mathbf{e}_z = \rho \mathbf{a}$$



In the limit,  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$  approach zero we get

$$\rho \mathbf{g} = \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \left[ \frac{P|_{x+\Delta x} - P|_x}{\Delta x} \mathbf{e}_x - \frac{P|_{y+\Delta y} - P|_y}{\Delta y} \mathbf{e}_y - \frac{P|_{z+\Delta z} - P|_z}{\Delta z} \mathbf{e}_z \right] + \rho \mathbf{a}$$

or

$$\rho \mathbf{g} = \frac{\partial P}{\partial x} \mathbf{e}_x + \frac{\partial P}{\partial y} \mathbf{e}_y + \frac{\partial P}{\partial z} \mathbf{e}_z + \rho \mathbf{a} \quad (4.4)$$

or

$$\rho \mathbf{g} = \nabla P + \rho \mathbf{a} \quad (4.5)$$

or

$$\nabla P = \rho(\mathbf{g} - \mathbf{a}) \quad (4.6)$$

where  $\nabla P \equiv \frac{\partial P}{\partial x} \mathbf{e}_x + \frac{\partial P}{\partial y} \mathbf{e}_y + \frac{\partial P}{\partial z} \mathbf{e}_z$  is called the gradient of  $P$ . For the *inertial* frame of reference setting  $\mathbf{a} = 0$  gives

$$\nabla P = \rho \mathbf{g} \quad (4.7)$$

Equation (4.7) is the basic equation of fluid statics and states that the maximum rate of change of pressure occurs in the direction of the gravitational vector. In addition, since isolines are perpendicular to the gradient, constant pressure lines are perpendicular to the gravitational vector. The point-to-point variation in pressure may be obtained by integrating Equation (4.7).

## Incompressible Fluid

For most engineering applications the variation in  $g$  is negligible, so our main concern is with the possible variation in the fluid density. For incompressible liquids the variation in density is usually negligible, even over large vertical distances, so that the assumption of constant density when dealing with liquids is a good one. Thus for these liquids Equation (4.7) gives

$$\frac{dP}{dy} \mathbf{e}_y = -\rho g \mathbf{e}_y \quad (4.8)$$

note that the negative sign appears due the fact that  $g$  is acting in opposite to  $y$  direction ( $g = -ge_y$ ) as indicated in Figure 4.8. Integrating the above equation

$$\int_{P_1}^{P_2} dP = -\rho g \int_{y_1}^{y_2} dy$$

$$P_1 - P_2 = \rho g(y_2 - y_1)$$

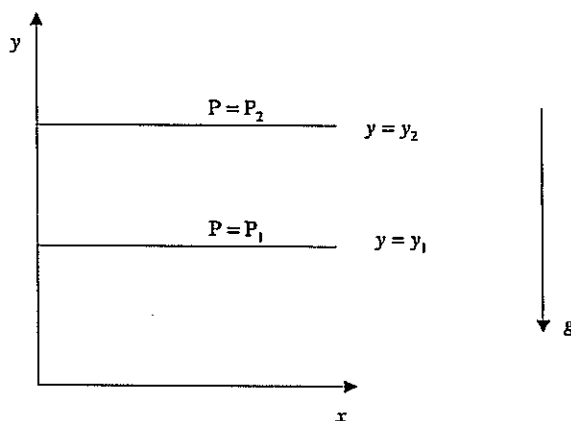


Figure 4.8: Notation for pressure variation in a fluid at rest.

## Compressible Fluid

We normally think of gases such as air, oxygen, and nitrogen as being compressible fluids since the density of the gas can change significantly with changes in pressure and temperature. Thus it is necessary to consider the possible variation in  $\rho$  before the Equation (4.8) can be integrated.

For those situations in which the variations in heights are large, attention must be

given to the variation in  $\rho$ . The equation of state for an ideal (or perfect) gas is

$$P = \rho RT$$

where  $P$  is the absolute pressure,  $R$  is the gas constant, and  $T$  is the absolute temperature. The relationship can be combined with Equation (4.8) to give

$$\frac{dP}{dy} = -\frac{gP}{RT}$$

and by separating variables

$$\int_{P_1}^{P_2} \frac{dP}{P} = \ln \frac{P_2}{P_1} = -\frac{g}{R} \int_{y_1}^{y_2} \frac{dy}{T}$$

where  $g$  and  $R$  are assumed to be constant over the elevation change from  $y_1$  to  $y_2$ . Before completing the integration, one must specify the nature of the variation of temperature with elevation. For example, if we assume that the temperature has a constant value  $T_0$  over the range  $y_1$  to  $y_2$  (*isothermal* conditions), then above equation becomes

$$P_2 = P_1 \exp \left[ -\frac{g(y_2 - y_1)}{RT_0} \right] \quad (4.9)$$

### Example 3:

Because of a leak in a buried gasoline storage tank, water has seeped into the depth shown in Figure 4.9. If the specific gravity of the gasoline is  $SG=0.68$ , determine the pressure at the gasoline-water interface, and at the bottom of the tank.

Since we are dealing with liquids at rest, the pressure distribution will be hydrostatic, and therefore the pressure variation can be found from the equation:

$$P = \rho gh + P_0$$

where  $P_0$  corresponds to the pressure at the free surface of the gasoline, then the pressure at the interface is

$$\begin{aligned} P_1 &= SG\rho_{\text{water}}gh + P_0 \\ &= (0.68)(62.4)(17) + P_0 \\ &= 721 + P_0 \text{ lb/ft}^2 \end{aligned}$$

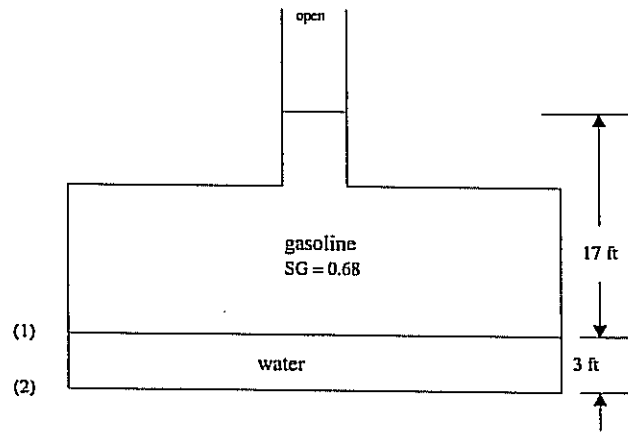


Figure 4.9: Gasoline storage tank.

If we measure the pressure relative to atmospheric pressure (gage pressure), it follows that  $P_0 = 0$ , and therefore

$$P_1 = 721 \text{ lb/ft}^2$$

We can now apply the same relationship to determine the pressure at the tank bottom:

$$\begin{aligned} P_2 &= \rho_{\text{water}} g h_{\text{water}} + P_1 \\ &= (62.4)(3) + 721 \\ &= 908 \text{ lb/ft}^2 \end{aligned}$$

Observe that if we wish to express these pressures in terms of *absolute* pressure, we would have to add the local atmospheric pressure (in appropriate units) to the previous results.

#### Example 4: Inclined-Tube Manometer

To measure small pressure changes, a manometer of the type shown in Figure 4.10 is frequently used. One leg of the manometer is inclined at an angle  $\theta$ , and the differential

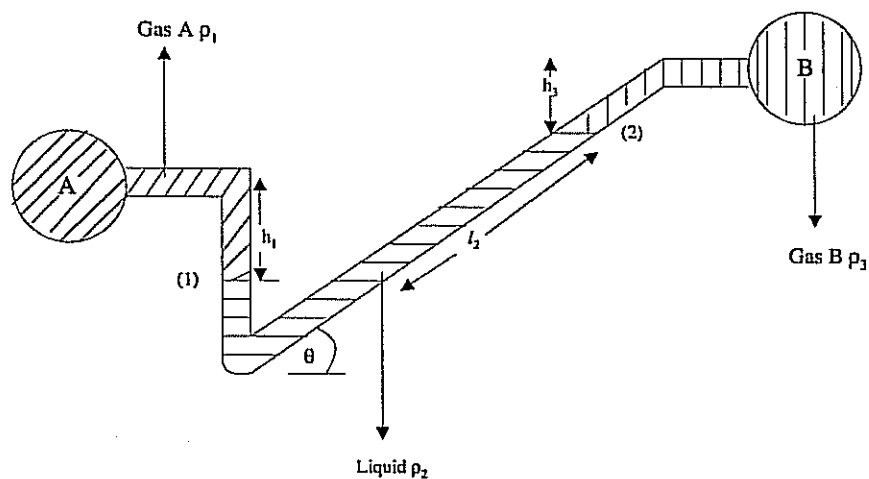


Figure 4.10: Inclined-tube manometer.

reading  $l_2$  is measured along the inclined tube. The difference in pressure  $P_A - P_B$  can be expressed as

$$P_A + \rho_1 g h_1 - \rho_2 g l_2 \sin \theta - \rho_3 g h_3 = P_B$$

or

$$P_A - P_B = \rho_2 g l_2 \sin \theta + \rho_3 g h_3 - \rho_1 g h_1$$

where it is to be noted that pressure difference between points (1) and (2) is due to the *vertical* distance between the points, which can be expressed as  $l_2 \sin \theta$ . Thus, for relatively small angles the differential reading along the inclined tube can be made large even for small pressure differences. The inclined-tube manometer is often used to measure small differences in gas pressures so that if pipes A and B contain a gas then

$$P_A - P_B = \rho_2 g l_2 \sin \theta$$

or

$$l_2 = \frac{P_A - P_B}{\rho_2 g \sin \theta}$$

where the contributions of the gas columns  $h_1$  and  $h_2$  have been neglected. The above equation shows that the differential reading  $l_2$  (for a given pressure difference) of the inclined-tube manometer can be increased over that obtained with a conventional U-tube manometer by the factor  $1/\sin \theta$ .

## 4.6 Forces on Submerged Surfaces

Determination of the forces on submerged surfaces is done frequently in fluid statics. Since these forces are due to pressure, use will be made of the relations describing the point-to-point variation in pressure which have been developed in the previous sections. The plane surface illustrated in Figure 4.11 is inclined at an angle  $\alpha$  to the surface of the fluid. The area of the inclined plane is  $A$ , and the density of the fluid is  $\rho$ .

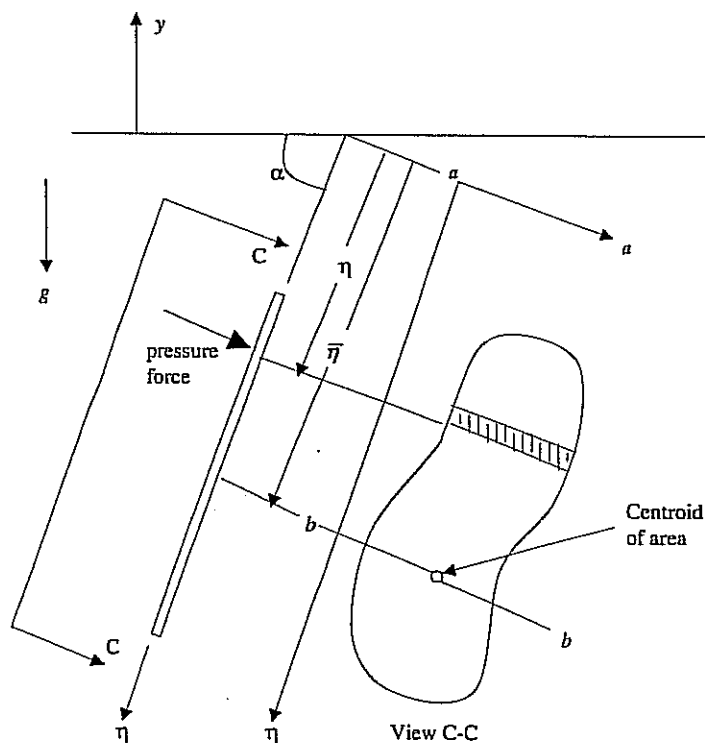


Figure 4.11: A submerged plane surface.

The magnitude of the force  $dF$  on the element  $dA$  is

$$dF = (P) dA = (\rho g \eta \sin \alpha) dA$$

Thus the total force  $F$  on the surface is obtained by the integration of the above equation

$$F = \int_A dF = \rho g \sin \alpha \int_A \eta dA$$

The definition of the centroid of the area is

$$\bar{\eta} \equiv \frac{1}{A} \int_A \eta dA$$

and thus

$$F = \rho g \sin \alpha \bar{\eta} A$$

Thus the force due to the pressure is equal to the pressure evaluated at the centroid of the submerged area multiplied by the submerged area. The point at which this force acts (the center of pressure) is not the centroid of the area. In order to find the center of pressure, we must find the point at which the total force on the plane must be concentrated in order to produce the same moment as the distributed pressure, or

$$F \eta_{c.p.} = \int_A \eta (\rho g \eta \sin \alpha) dA$$

and since  $F = \rho g \sin \alpha \bar{\eta} A$ , we have

$$\eta_{c.p.} = \frac{1}{A \bar{\eta}} \int_A \eta^2 dA = \frac{I_{aa}}{A \bar{\eta}}$$

The moment of the area about the surface may be translated from an axis  $aa$  located at the fluid surface to an axis  $bb$  through the centroid by

$$I_{aa} = I_{bb} + \bar{\eta}^2 A$$

and thus

$$\eta_{c.p.} - \bar{\eta} = \frac{I_{bb}}{A \bar{\eta}}$$

The center of pressure is located below the centroid a distance  $I_{bb}/A\bar{\eta}$ .