Chapter 6

Macroscopic Balances

6.1 Conservation of Mass

The integral expression for the mass balance over a general control volume is given by

$$\iint_{\text{c.s.}} \rho(\mathbf{v}\cdot\mathbf{n})\ dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho\ dV = 0 \tag{6.1}$$

where the first integration is carried out over the control surface and it represents the net rate of mass efflux from the control volume. The second integration is carried out over the control volume and it represents the rate of accumulation of mass within the control volume. ρ is the density of fluid flowing through differential control volume dV and leaving/entering through a differential control surface dA with velocity $\mathbf{v}$ where the (outward) unit normal to the surface is $\mathbf{n}$.

Note that $\mathbf{v}\cdot\mathbf{n} = |\mathbf{v}||\mathbf{n}|\cos\theta$ where θ is the angle between the velocity vector, $\mathbf{v}$, and the *outward* directed unit normal vector, $\mathbf{n}$, to dA. Thus if both $\mathbf{v}$ and $\mathbf{n}$ are in the same direction ($\theta = 0$) then $\mathbf{v}\cdot\mathbf{n} = |\mathbf{v}||\mathbf{n}| = |\mathbf{v}|$ and if $\mathbf{v}$ and $\mathbf{n}$ are facing in the opposite direction ($\theta = 180$) then $\mathbf{v}\cdot\mathbf{n} = -|\mathbf{v}||\mathbf{n}| = -|\mathbf{v}|$, where $|\mathbf{v}|$ is simply the magnitude of the velocity vector at that location.

If flow is steady relative to coordinates fixed to the control volume, then the accumulation term will be zero. Thus, for this situation the above equation reduces to

$$\iint_{\text{c.s.}} \rho(\mathbf{v}\cdot\mathbf{n})\,dA = 0 \tag{6.2}$$

Consider the steady one-dimensional flow into and out of a control volume as shown in Figure 6.1.

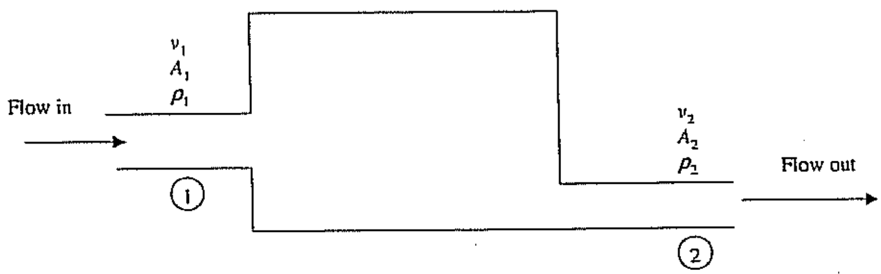

Figure 6.1: Steady one-dimensional flow into and out of a control volume.

In this case the above equation becomes

$$\iint_{\text{c.s.}} \rho(\mathbf{v}\cdot\mathbf{n})\,dA = \iint_{A_1} \rho(\mathbf{v}\cdot\mathbf{n})\,dA + \iint_{A_2} \rho(\mathbf{v}\cdot\mathbf{n})\,dA = 0$$

or

$$\iint_{\text{c.s.}} \rho(\mathbf{v}\cdot\mathbf{n})\,dA = -\iint_{A_1} \rho v\,dA + \iint_{A_2} \rho v\,dA = 0$$

or simply

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 \tag{6.3}$$

If the fluid is incompressible with a constant density ρ then the above equation becomes

$$v_1 A_1 = v_2 A_2 \tag{6.4}$$

Example-1: Consider a tank having cross sectional area A initially filled with a fluid of density ρ upto height h_0 as shown in Figure 6.2. At $t = 0$ the bottom tube having cross sectional area a is opened. Determine how the height, $h(t)$, of fluid in the tank changes with time.

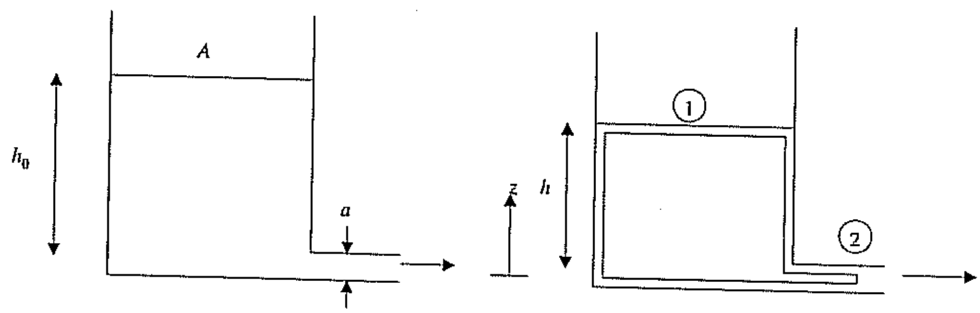

Figure 6.2: Flow in a tank opened at the bottom.

Writing B.E. between points (1) and (2) as shown in the control volume

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2$$

In this problem $p_1 = p_2 = p_{atm}$, $z_1 = h(t)$, $z_2 = 0$ and $v_1 << v_2$ which is a valid assumption if $a << A$. Thus we can set $v_1 \approx 0$. Then the above equation gives

$$v_2 = \sqrt{2gh(t)}$$

Applying Equation (6.1) for the control volume shown we get

$$\iint_{\text{c.s.}} \rho(\mathbf{v}\cdot\mathbf{n})\, dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho\, dV = 0$$

$$-\iint_{A_1} \rho v\, dA + \iint_{A_2} \rho v\, dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho\, dV = 0$$

$$\iint_{A_2} v\, dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} dV = 0$$

$$v_2(a) + \frac{d}{dt}(V) = 0$$

$$\sqrt{2gh(t)}(a) + \frac{d}{dt}(Ah(t)) = 0$$

$$a\sqrt{2g}\sqrt{h} + A\frac{dh}{dt} = 0$$

$$\frac{dh}{dt} = -\frac{a}{A}\sqrt{2g}\sqrt{h}$$

$$\frac{dh}{\sqrt{h}} = -\alpha\ dt$$

where $\alpha = \frac{a}{A}\sqrt{2g}$. Integrating the above equation between the limits: $t = 0, h = h_0$ and $t = t, h = h(t)$ we get

$$\int_{h_0}^{h(t)} \frac{dh}{\sqrt{h}} = -\alpha \int_0^t dt$$

$$2\left[\sqrt{h}\right]_{h_0}^{h(t)} = -\alpha\ t$$

$$\sqrt{h(t)} - \sqrt{h_0} = -\frac{\alpha}{2}\ t$$

or

$$\sqrt{h(t)} = \sqrt{h_0} - \frac{\alpha}{2}\cdot t$$

The above expression is the desired expression. From this we can see that the height of fluid decreases from initial height h_0 as time passes and eventually the tank is completely empty. We can determine the corresponding time, t_f, by setting $h(t) = 0$ in the above expression. This gives

$$t_f = \frac{2}{\alpha}\sqrt{h_0}$$

where $\alpha = \frac{a}{A}\sqrt{2g}$.

Example-2:

A tank with a square base of side S is open to the atmosphere and it is being filled with water through a pipe of diameter d in which the water velocity is v_{in}. Water seeps through the porous bottom of the tank with velocity v_0, given by the equation

$$v_0 = \frac{\kappa}{\mu}(p - p_a)$$

where κ, μ and p denote permeability of the tank bottom, viscosity of the water and pressure inside the tank at the bottom, respectively. Derive an expression to describe the variation of water level in the tank, h, with time, if the level at time $t = 0$ is h_0. For the purpose of calculating pressure p, liquid in the tank may be considered static.

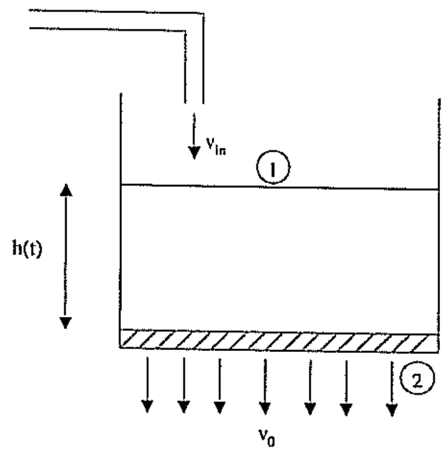

Figure 6.3: Flow in a tank with porous bottom.

Applying Equation (6.1) for the control volume shown we get

$$\iint_{\text{c.s.}} \rho(\mathbf{v}\cdot\mathbf{n})\, dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho\, dV = 0$$

$$-\iint_{A_1} \rho v\, dA + \iint_{A_2} \rho v\, dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho\, dV = 0$$

$$-v_{\text{in}}\frac{\pi}{4}d^2 + \frac{\kappa}{\mu}(p - p_a)S^2 + \frac{d(S^2 h)}{dt} = 0$$

Note that the fluid is incompressible hence ρ is treated as a constant. Pressure p at the bottom of the tank at any time t is given by

$$p = \rho g h(t) + p_a$$

Substituting this in the above equation and rearranging gives

$$\frac{dh}{dt} = \frac{\pi}{4}\left(\frac{d}{S}\right)^2 v_{\text{in}} - \left(\frac{\kappa}{\mu}\rho g\right) h$$

$$\frac{dh}{dt} = \alpha(\beta - h) \tag{6.5}$$

where

$$\alpha = \frac{\kappa}{\mu}\rho g, \quad \beta = \frac{\pi}{4}\left(\frac{d}{S}\right)^2 v_{\text{in}}\frac{\mu}{\kappa\rho g} \tag{6.6}$$

Separation of variables in Equation (6.5) gives

$$\frac{dh}{\beta - h} = \alpha dt$$

which upon integration between the limits from $t = 0, h = h_0$ to $t = t, h = h(t)$ gives

$$-\ln\left[\frac{\beta - h}{\beta - h_0}\right] = \alpha t$$

which becomes

$$\beta - h = (\beta - h_0)e^{-\alpha t}$$

$$h(t) = \beta - (\beta - h_0)e^{-\alpha t}$$

This is the desired expression for $h(t)$ where α and β are given by Equation (6.6).

6.2 Conservation of Momentum

The integral expression for the linear-momentum balance over a general control volume is given by

$$\sum \mathbf{F} = \iint_{\text{c.s.}} \mathbf{v}\rho(\mathbf{v}\cdot\mathbf{n})\; dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho\mathbf{v}\; dV \tag{6.7}$$

where $\sum \mathbf{F}$ represents the total force acting on the control volume. The total force acting on the control volume consists both of surface forces due to interactions between the control-volume fluid, and its surroundings through direct contact, and of body forces resulting from the location of the control volume in a force field. The gravitational field and its resultant force are the most common example of this later type. The first integration on the right side in Equation (6.7) is carried out over the control surface and it represents the net rate of momentum efflux from the control volume. The second integration is carried out over the control volume and it represents the rate of accumulation of linear momentum within the control volume. ρ is the density of fluid flowing through differential control volume dV and leaving/entering through a differential control surface dA with velocity $\mathbf{v}$ where the (outward) unit normal to the surface is $\mathbf{n}$.

Note that $\mathbf{v}\cdot\mathbf{n} = |\mathbf{v}||\mathbf{n}|\cos\theta$ where θ is the angle between the velocity vector, $\mathbf{v}$, and the *outward* directed unit normal vector, $\mathbf{n}$, to dA. Thus if both $\mathbf{v}$ and $\mathbf{n}$ are in the same direction ($\theta = 0$) then $\mathbf{v}\cdot\mathbf{n} = |\mathbf{v}||\mathbf{n}| = |\mathbf{v}|$ and if $\mathbf{v}$ and $\mathbf{n}$ are facing in the opposite direction ($\theta = 180$) then $\mathbf{v}\cdot\mathbf{n} = -|\mathbf{v}||\mathbf{n}| = -|\mathbf{v}|$, where $|\mathbf{v}|$ is simply the magnitude of the velocity vector at that location.

Note that Equation (6.7) is a vector equation as opposed to Equation (6.1) which is a scalar equation. In rectangular coordinates the single-vector equation (6.7) may be written as three scalar equations

$$\sum F_x = \iint_{\text{c.s.}} v_x \rho(\mathbf{v}\cdot\mathbf{n})\; dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho v_x\; dV \tag{6.8}$$

$$\sum F_y = \iint_{\text{c.s.}} v_y \rho(\mathbf{v}\cdot\mathbf{n})\; dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho v_y\; dV \tag{6.9}$$

$$\sum F_z = \iint_{\text{c.s.}} v_z \rho(\mathbf{v}\cdot\mathbf{n})\; dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho v_z\; dV \tag{6.10}$$

When applying any or all of the above equations, it must be remembered that each term has a sign with respect to the positively defined x, y, and z directions. The determination of the sign of the surface integral should be considered with special care, since both the velocity vector component (v_x) and the scalar product ($\mathbf{v}\cdot\mathbf{n}$) have signs as explained before. The combination of the proper sign associated with each of these terms will give the correct sense of the integral. It should also be remembered that since Equations (6.8) to (6.10) are written for the fluid in the control volume, *the forces to be employed in these equations are those acting on the fluid.*

If flow is steady relative to coordinates fixed to the control volume, then the accumulation term will be zero. Thus, for this situation the Equation (6.7) reduces to

$$\sum \mathbf{F} = \iint_{c.s.} \mathbf{v}\rho(\mathbf{v}\cdot\mathbf{n})\ dA \tag{6.11}$$

Example-3:

Water flows steadily through the horizontal 30^0 pipe bend as shown in Figure 6.4. At station 1 the diameter is 0.3 m, the velocity is 12 m/s, and the pressure is 128 kPa gage. At station 2 the diameter is 0.38 m and the pressure is 145 kPa gage. Determine the forces F_x and F_y necessary to hold the pipe bend stationary.

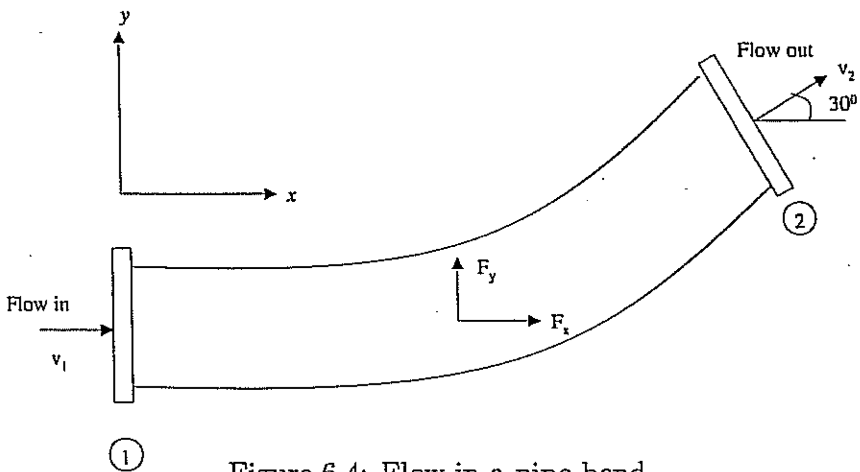

Figure 6.4: Flow in a pipe bend.

$$A_1 = \frac{\pi}{4}D_1^2 = 0.0707\ \text{m}^2$$
$$A_2 = \frac{\pi}{4}D_2^2 = 0.1134\ \text{m}^2$$

Using Equation (6.4) one can determine velocity v_2

$$\begin{aligned} v_2 &= v_1\frac{A_1}{A_2} \\ &= 12\frac{0.0707}{0.1134} \\ &= 7.48\ \text{m/s} \end{aligned}$$

At steady state Equation (6.8) gives

$$\begin{aligned}
F_x &+ P_1A_1 - P_2A_2\cos\theta = v_1\rho(-v_1)A_1 + v_2\rho(v_2\cos\theta)A_2 \\
F_x &= -P_1A_1 + P_2A_2\cos\theta - v_1\rho(v_1)A_1 + v_2\rho(v_2\cos\theta)A_2 \\
&= -(128\times10^3)(0.0707) + (145\times10^3)(0.1134)\cos 30 \\
&\quad -(12)(10^3)(12)(0.0707) + (7.48)(10^3)(7.48\cos 30)(0.1134) \\
&= 505.5 \text{ N}
\end{aligned}$$

Similarly, at steady state Equation (6.9) gives

$$\begin{aligned}
F_y &- W - P_2A_2\sin\theta = v_2\rho(v_2\sin\theta)A_2 \\
F_y &= W + P_2A_2\sin\theta + v_2\rho(v_2\sin\theta)A_2 \\
&= W + (145\times10^3)(0.1134)\sin 30 + (7.48)(10^3)(7.48\sin 30)(0.1134) \\
&= W + 11395 \text{ N}
\end{aligned}$$

where W is the weight of water contained within bend.

6.3 Conservation of Energy

The integral expression for the conservation of energy over a general control volume is given by

$$\frac{\delta Q}{dt} - \frac{\delta W_s}{dt} = \iint_{\text{c.s.}} \left(e + \frac{P}{\rho}\right)\rho(\mathbf{v}\cdot\mathbf{n})\,dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} e\rho\,dV + \frac{\delta W_\mu}{dt} \tag{6.12}$$

where δQ and δW_s represent differential heat transfer and the shaft work which is that done by the control volume on its surroundings that could cause a shaft to rotate or accomplish the raising of a weight through a distance, respectively. δQ is positive when heat is added to the system, δW_s is positive when work is done by the system.

The quantity e is the specific energy or the energy per unit mass. The specific energy includes the potential energy, gy, due to the position of the fluid continuum in the gravitational field; the kinetic energy of the fluid, $v^2/2$, due to its velocity; and the internal energy, u, of the fluid due to its thermal state.

Term $\delta W_\mu/dt$, represents the work rate accomplished in overcoming viscous effects at the control surface. Note that this constitutes both the work associated with the viscous portion of the normal stress and the shear work.

Example-4: Consider the system shown in Figure 6.1 under the conditions of steady fluid flow and no frictional losses. For the specified conditions the overall energy expression, Equation (6.12) becomes

$$\frac{\delta Q}{dt} - \frac{\delta W_s}{dt} = \iint_{\text{c.s.}} \left(e + \frac{P}{\rho}\right) \rho(\mathbf{v}\cdot\mathbf{n})\, dA \tag{6.13}$$

The specific energy, e, may be expanded to include the kinetic, potential, and internal energy contributions to give

$$e + \frac{P}{\rho} = gy + \frac{v^2}{2} + u + \frac{P}{\rho}$$

Thus the surface integral becomes

$$\begin{aligned}\iint_{\text{c.s.}} \left(e + \frac{P}{\rho}\right) \rho(\mathbf{v}\cdot\mathbf{n})\, dA &= \left[\frac{v_2^2}{2} + gy_2 + u_2 + \frac{P_2}{\rho_2}\right](\rho_2 v_2 A_2) \\ &- \left[\frac{v_1^2}{2} + gy_1 + u_1 + \frac{P_1}{\rho_1}\right](\rho_1 v_1 A_1)\end{aligned}$$

The energy expression for this example now becomes

$$\begin{aligned}\frac{\delta Q}{dt} - \frac{\delta W_s}{dt} &= \left[\frac{v_2^2}{2} + gy_2 + u_2 + \frac{P_2}{\rho_2}\right](\rho_2 v_2 A_2) \\ &- \left[\frac{v_1^2}{2} + gy_1 + u_1 + \frac{P_1}{\rho_1}\right](\rho_1 v_1 A_1)\end{aligned} \tag{6.14}$$

Equation (6.3) says

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 = \dot{m}$$

where $\dot{m}$ is the mass flow rate. If each term in Equation (6.14) is divided by $\dot{m}$, we have

$$\frac{q - \dot{W}_s}{\dot{m}} = \left[\frac{v_2^2}{2} + gy_2 + u_2 + \frac{P_2}{\rho_2}\right] - \left[\frac{v_1^2}{2} + gy_1 + u_1 + \frac{P_1}{\rho_1}\right]$$

or, in more familiar form

$$\frac{v_1^2}{2} + gy_1 + h_1 + \frac{q}{\dot{m}} = \frac{v_2^2}{2} + gy_2 + h_2 + \frac{\dot{W}_s}{\dot{m}} \tag{6.15}$$

where the sum of the internal energy and flow energy, $u+P/\rho$, has been replaced by the enthalphy, h, which is equal to the sum of these quantities by definition $h \equiv u + P/\rho$.

Example-5: A shaft is rotating at constant angular velocity ω in the bearing shown in Figure 6.5. The shaft diameter is d and the shear stress acting on the shaft is τ. Find the rate at which energy must be removed from the bearing in order that the lubricating oil between the rotating shaft and the stationary bearing surface remains at constant temperature. The shaft is assumed to be lightly loaded and concentric with the journal.

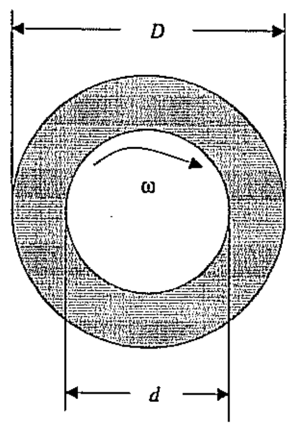

Figure 6.5: Bearing and control volume for bearing analysis.

The control volume selected consists of a unit length of the fluid surrounding the shaft as shown in Figure 6.5. From the figure we may observe the following:

- No fluid crosses the control surface.
- No shaft work crosses the control surface.
- The flow is steady.

Under these conditions Equation (6.12) reduces to

$$\frac{\delta Q}{dt} = \frac{\delta W_\mu}{dt}$$

Note that δW_μ constitutes both the work associated with the viscous portion of the normal stress, δW_σ, and the shear work, δW_τ. In this example there is no contribution from the normal stress. Hence, $\delta W_\mu = \delta W_\tau$. In this case all of the viscous work is done to overcome shearing stresses; thus the viscous work is $= \iint_{\text{c.s.}} \tau(\mathbf{v} \cdot \mathbf{e}_t)dA$. At the outer boundary, $v = 0$ and at the inner boundary $\iint_{\text{c.s.}} \tau(-v)(A) = -\tau(\omega d/2)\pi d$, where $\mathbf{e}_t$ indicates the sense of the shear stress, τ, on the surroundings. The resulting sign is consistent with the concept of work being positive when done by a system on its surrounding. Thus

$$\frac{\delta Q}{dt} = -\tau \frac{\omega d^2 \pi}{2}$$

which is the heat transfer rate required to maintain the oil at a constant temperature. If energy is not removed from the system then $\delta Q/dt = 0$, and

$$\frac{\partial}{\partial t} \iiint_{\text{c.v}} e\rho \, dV = -\frac{\delta W_\mu}{dt}$$

As only the internal energy of the oil will increase with respect to time

$$\frac{\partial}{\partial t} \iiint_{\text{c.v}} e\rho \, dV = \pi\rho\left(\frac{D^2 - d^2}{4}\right)\frac{du}{dt} = -\frac{\delta W_\mu}{dt} = \omega\frac{\pi d^2}{2}\tau$$

or, with constant specific heat c

$$c\frac{dT}{dt} = \frac{2\tau\omega d^2}{\rho(D^2 - d^2)}$$

where D is the outer bearing diameter.

In this example the use of the viscous-work term has been illustrated. Note that

- The viscous-work term involves only quantities on the surface of the control volume.
- When the velocity on the surface of the control volume is zero, the viscous-work term is zero.

6.4 The Bernoulli Equation

Under certain flow conditions Equation (6.12) applied to a control volume reduces to an extremely useful relation known as the Bernoulli equation. Consider the control volume shown in Figure 6.6 in which flow is steady, incompressible, and inviscid, and in which no heat transfer or change in internal energy occurs. Under these conditions Equation

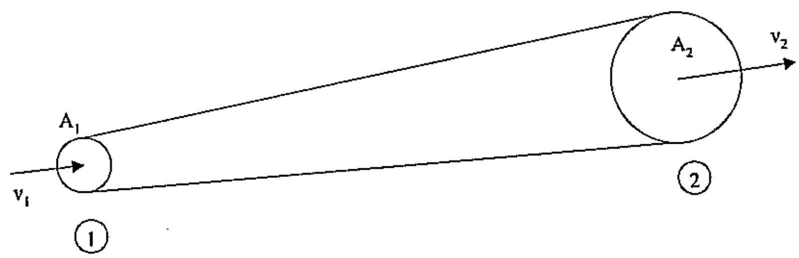

Figure 6.6: Control volume for steady, incompressible, inviscid, isothermal flow.

(6.12) reduces to

$$0 = \iint_{\text{c.s.}} \left(e + \frac{P}{\rho}\right) \rho(\mathbf{v} \cdot \mathbf{n})\, dA$$

or

$$0 = \left(gy_2 + \frac{v_2^2}{2} + \frac{P_2}{\rho}\right)(\rho v_2 A_2) - \left(gy_1 + \frac{v_1^2}{2} + \frac{P_1}{\rho}\right)(\rho v_1 A_1)$$

From Equation (6.4)

$$v_1 A_1 = v_2 A_2$$

which may be divided through to give

$$gy_1 + \frac{v_1^2}{2} + \frac{P_1}{\rho} = gy_2 + \frac{v_2^2}{2} + \frac{P_2}{\rho} \tag{6.16}$$

Dividing through by g, we have

$$y_1 + \frac{v_1^2}{2g} + \frac{P_1}{\rho g} = y_2 + \frac{v_2^2}{2g} + \frac{P_2}{\rho g} \tag{6.17}$$

Either of the above equations is designated the Bernoulli equation.
Note that each term in Equation (6.17) has the unit of length. The quantities are often designated "heads" due to elevation, velocity, and pressure, respectively. These terms, both individually and collectively, indicate the quantities which may be directly converted to produce mechanical energy.
Note that Bernoulli equation has the following limitations

- Inviscid flow.
- Steady flow.
- Incompressible flow.
- The equation applies along a streamline.